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Xuan Li, Dunant Halim, Xiaoling Liu

University of Nottingham
UK | CHINA | MALAYSIA

University of Nottingham Ningbo China, 199 Taikang East Road, Ningbo, 315100, Zhejiang, China.

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Xuan Li[1]
Department of Mechanical, Materials and Manufacturing Engineering
The University of Nottingham Ningbo China
Ningbo, Zhejiang, China 315100

Dunant Halim[2*]
Department of Mechanical, Materials and Manufacturing Engineering
The University of Nottingham Ningbo China
Ningbo, Zhejiang, China 315100

Xiaoling Liu[3]
Department of Mechanical, Materials and Manufacturing Engineering
The University of Nottingham Ningbo China
Ningbo, Zhejiang, China 315100

ABSTRACT

This work aims to study forced vibration characteristics of Fibre-Reinforced Polymer (FRP) composite laminated beam with different properties, through a development of an analytical model using the Green function method. The forced vibration characteristics of a FRP laminated beam structure is generally more complex than those of a homogeneous beam structure since each layer is anisotropic with a different layer having different properties. In this work, the Green function method is used to model an FRP laminated beam to solve the associated equation of motion. The proposed analytical model allows a more efficient parametric analysis to be done on FRP laminated beams, in contrast to using a numerical model that is more computationally expensive. The analytical model is verified through a comparison with the numerical model of FRP laminated beam. Based on the developed model, a FRP laminated beam with various fibre orientations, is studied under forced vibration, demonstrating the effectiveness of the proposed method for forced vibration analysis of a laminated beam.

1. INTRODUCTION

[1] xuan.li@nottingham.edu.cn
[2*] dunant.halim@ nottingham.edu.cn (Corresponding author)
[3] xiaoling.liu@ nottingham.edu.cn

Composite laminate beams have the advantages of lightweight, high modulus and good corrosion resistance, and they are therefore widely used in many engineering applications. Hence, it is important to be able to investigate the vibration characteristics of composite laminate beams with different designs for a variety of applications. Previous research works have investigated analytical modelling for free vibration of composite laminate beam for different applications [1-3], but there are more limited works focused on the forced vibration of composite laminate beams.

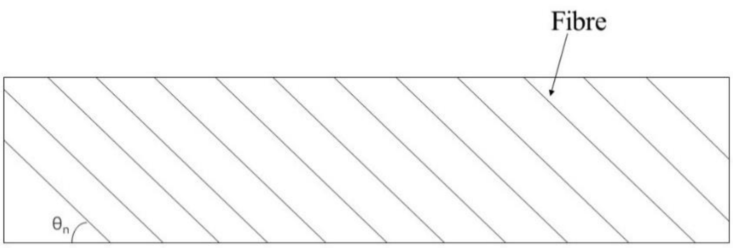
There are a number of analytical modelling methods that have been developed for forced vibration analysis of beam structures. One of the most commonly used methods for modelling forced vibration is to represent the structural dynamic response in terms of the associated eigenfunctions [4]. The method utilises a solution in the form of the summation of an infinite series, which needs to be truncated down to a certain number of terms in calculation, causing a truncation error. There are other methods, such as the double Laplace transform method and the method of expansion in eigenforms, which can provide exact solutions although the solutions for obtaining forced vibration response tend to be more complicated [5-6].

On the other hand, the Green function method has a generally reliable performance for forced vibration analysis if compared with the previous methods because it has a relatively simple formulation whilst providing exact solutions. Abu-Hilal demonstrated the feasibility of the Green function method for modelling forced vibration of beam structures with different boundary conditions and applied excitations [6]. Kukla et al. demonstrated the effectiveness of the Green function method for analysing vibration response of stepped beams with axial load [7]. Li et al. used the Green function method to solve vibration response of Timoshenko beams with damping effect [8]. In addition, the Green function method has also been used by other researchers to analyse coupled dynamic problems of beams such as for thermoelastic coupling vibration [9-10]. The accuracy of the Green function method in solving forced vibration problems have been demonstrated [7-8, 11], although they are not focused on analysing the forced vibration of a composite laminate beam. Hence, this work intends to fill this gap by developing a model for forced vibration of an FRP laminate beam that is based on the Green function method.

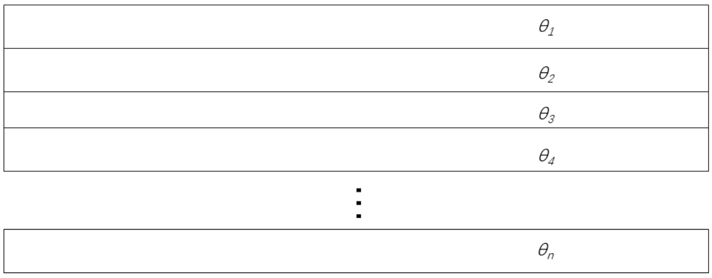
2. MODELLING OF A COMPOSITE LAMINATE BEAM

2.1. Composite laminate beams:

The structure of a composite laminate beam composed by fibres and polymer resin is illustrated in Figure 1, with the angle of fibre orientation in each layer denoted as θ_n.



(a) Top View



(b) Side View

Figure 1 The structure of a composite laminate beam.

Based on the Euler-Bernoulli beam theory, the equation of motion for the composite laminate beam can be described by:

$$E\frac{\partial^4 u}{\partial x^4} + \rho A\frac{\partial^2 u}{\partial t^2} = p(x,t) \tag{1}$$

where E is the equivalent bending stiffness of the beam; ρ is the density and A is the cross-sectional area. Based on the Classical Lamination Theory (CLT), the equivalent bending stiffness of beam E can be calculated by:

$$E = D_{11} - B_{11}/A_{11}. \tag{2}$$

Here, D_{11} is bending stiffness; B_{11} is the bending-extension coupling stiffness and A_{11} is the extension stiffness of laminate beam which is related to the fibre angle θ_n. The detailed derivation process can be found in reference [12]. In this case, $p(x,t)$ is the driving force and it can be defined as:

$$p(x,t) = p(x)\, sin(\omega t). \tag{3}$$

Meanwhile, the structural displacement at location *x* over the beam can be described as:

$$u(x,t) = u(x)\, sin\, \omega\, t. \tag{4}$$

Substituting equations (3) and (4) into equation (1), the following equation can be obtained:

$$\frac{\partial^4 u}{\partial x^4} - k^4\frac{\partial^2 u}{\partial t^2} = f(x) \tag{5}$$

where $k^4 = \frac{\omega^2 \rho A}{E}$ and $f(x) = \frac{p(x)}{E}$.

2.2 The Green function

The Green function of the four sub-beams can be determined by using the following equations [6]:

$$\frac{\partial^4 G(x,\xi)}{\partial x^4} - k^4\frac{\partial^2 G(x,\xi)}{\partial t^2} = \frac{\delta(x-\xi)}{E} \tag{6}$$

where $\delta(x-\xi)$ is the Dirac function, and it has the following property:

$$\int_0^L P(\xi)\,\delta(x-\xi)d\xi = P(x). \tag{7}$$

The solution for the structural displacement in equation (4) can be obtained by incorporating equations (6) and (7) as follows:

$$\int_0^L P(\xi)\, G(x,\xi)d\xi = u(x). \tag{8}$$

This can then be solved by using the Laplace transform as:

$$\hat{G}(s,\xi) = \frac{1}{(s^4-k^4)}\left[\frac{e^{-s\xi}}{E} + s^3 G(0) + s^2 \dot{G}(0) + s\ddot{G}(0) + \dddot{G}(0)\right]. \tag{9}$$

By taking the inverse Laplace Transform, the Green function can be found as:

$$G(x,\xi) = \frac{\varphi_4(x-\xi)H(x-\xi)}{Ek^3} + G(0)\varphi_1(x) + \frac{\dot{G}(0)}{k}\varphi_2(x) + \frac{\ddot{G}(0)}{k^2}\varphi_3(x) + \frac{\dddot{G}(0)}{k^3}\varphi_4(x) \tag{10}$$

where:

$$\varphi_1(x) = \frac{1}{2}(cosh(kx) + cos(kx)) \tag{11a}$$

$$\varphi_2(x) = \frac{1}{2}(sinh(kx) + sin(kx)) \tag{11b}$$

$$\varphi_3(x) = \frac{1}{2}(cosh(kx) - cos(kx)) \tag{11c}$$

$$\varphi_4(x) = \frac{1}{2}(sinh(kx) - sin(kx)) \tag{11d}$$

In this study, a cantilever beam structure is investigated, which has the following boundary conditions:

$$u(0) = 0 \tag{12}$$

$$\dot{u}(0) = 0 \tag{13}$$

$$V(L) = -E\dddot{u}(L) = 0 \tag{14}$$

$$M = -E\ddot{u}(L) = 0. \tag{15}$$

Substituting equations (9) – (11) into equations (12) – (15), the following can be obtained:

$$T(\omega)X = S \tag{16}$$

where the $T(\omega)$ is a 4×4 matrix which is a function of angular frequency ω:

$$T(\omega) = \begin{bmatrix} \varphi_1(0) & \varphi_2(0)/k & \varphi_3(0)/k^2 & \varphi_4(0)/k^3 \\ k\varphi_4(0) & \varphi_1(0) & \varphi_2(0)/k & \varphi_3(0)/k^2 \\ -Ek^3\varphi_2(L) & -Ek^2\varphi_3(L) & -Ek\varphi_4(L) & -E\varphi_1(L) \\ -Ek^2\varphi_3(L) & -Ek\varphi_4(L) & -E\varphi_1(L) & -E\varphi_2(L)/k \end{bmatrix} \tag{17}$$

Here, X and S are described as:

$$X = [G(0) \quad \dot{G}(0) \quad \ddot{G}(0) \quad \dddot{G}(0)] \tag{18}$$

$$S = [-\frac{\varphi_4(x-\xi)H(x-\xi)}{Ek^3} \quad -\frac{\varphi_3(x-\xi)H(x-\xi)}{Ek^2} \quad \varphi_1(x-\xi)H(x-\xi) \quad \frac{\varphi_2(x-\xi)H(x-\xi)}{k}] \tag{19}$$

The solution to the forced vibration of a composite laminate beam can then be obtained by solving equation (16).

3. FREE AND FORCED VIBRATION OF A LAMINATE BEAM

In this section, a model of a symmetric FRP laminate beam is constructed with the parameters shown in Table 1. The laminate beam has 8 layers with stacking sequence as $[0/\theta_2]_{2s}$. Based on this configuration, $B_{11} = 0$ and equation (2) can be simplified as $E = D_{11}$.

Table 1: Parameters of the composite laminate beam structure.

Properties	**Symbol**	**Value**
Fibre modulus	E_1 (GPa)	134.49
Resin modulus	E_2 (GPa)	10.34
Shear modulus	G_{12} (GPa)	5
Poisson's ratio	υ_{12}	0.33
Density	ρ(kg/m^3)	1500
Length	L (m)	0.127
Height	h (mm)	1.016
Width	b (m)	0.0127

3.1. Free vibration

The free vibration of the laminate beam structure is investigated in this part, by setting $S = 0$. Equation (16) is expressed as:

$$T(\omega)X = 0. \tag{20}$$

Based on the non-trivial solution of X, the natural frequencies and mode shapes can be determined by solving equation:

$$\det(T(\omega)) = 0. \tag{21}$$

To verify the accuracy of the developed model, a similar composite laminate beam configuration used in [13-14] is utilised. Finite Element Model (FEM) for this beam configuration is also constructed and the results are compared in Table 2. It can be observed that the natural frequencies obtained from the developed model are generally consistent with those from the references [13-14] and FEM analysis.

Table 2: The first three natural frequencies (Hz) *of a composite laminate beam.*

Mode	**References [13-14]**	**FEM**	**Present Model**
1	79.83	81.51	81.61
2	515.26	511.91	511.49
3	1442.89	1440.00	1432.20

3.2. Forced vibration

For forced vibration case, considering the driving angular frequency $\omega = \omega_f$, equation (16) can be described as:

$$X = T^{-1}(\omega_f)S. \tag{22}$$

where $T^{-1}(\omega_f)$ represents the inverse matrix of $T(\omega_f)$. By solving vector X and utilising equation (10), the displacement for the laminate beam under the applied excitation can be obtained. In this work, the excitation is in the form of a concentrated harmonic force at location x_o as shown in following formulation:

$$P(x) = P_0\delta(x - x_0). \tag{23}$$

Therefore, the solution can be solved from equation (8) as follows:

$$\int_0^L P_0\delta(\xi - x_0)\, G(x,\xi)d\xi = P_0 G(x, x_0) \tag{24}$$

where $G(x, x_0)$ as a function of ω_f, is described by equations (10) and (11).

Consider an example where the driving frequency is set as $f = 1Hz$ (angular frequency $\omega_f = 2\pi\ rad/s$). The amplitude and the location of the driving force are $P_0 = 1$ N and $x_0 = 0.127$m (x_0=L, right end of cantilever beam), respectively as shown in Figure 2:

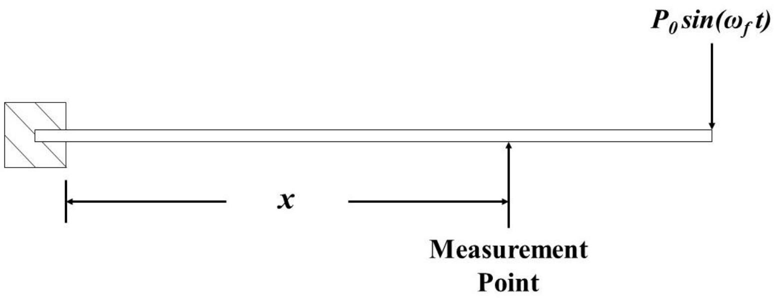


Figure 2. A composite laminate beam under a concentrated harmonic excitation.

The fibre angles θ_2 for different layers are set as 0, $\pi/6$, $\pi/4$ and $\pi/2$ respectively and the displacement response at the measurement point located at *x=0.108m (x=0.85L)* in time domain with different fibre angles are shown in Figure 3. A numerical model of these laminate beams is also constructed using the finite element method and the results obtained are compared with the analytical model in Figure 3(b).

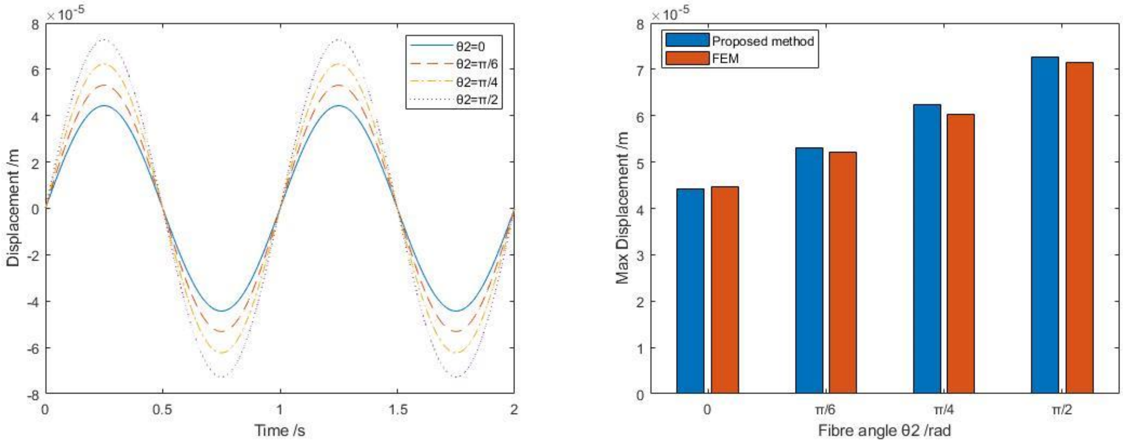


Figure 3. The forced vibration response of a composite laminate beam: (a) Time domain displacement responses. (b) Maximum displacements for different fibre angles.

It is observed from the figures that the structural displacement increases as the fibre angle θ_2 increases. This result can be expected because when the fibre angle θ_2 increases, the high modulus of fibres contributes less to the flexural stiffness of the composite laminate beam. It can also be seen from Figure 3(b) that the difference between the FEM and developed model's results is lower than 3.5%, so the model is relatively accurate for simulating a composite laminate beam under forced vibration.

4. CONCLUSIONS

An analytical model based on the Green function method has been developed for analysing the forced vibration response of composite FRP laminate beams. The accuracy and advantages of proposed method were demonstrated by comparing the results with those from the references and FEM for both free vibration and forced vibration cases. The work has also investigated the relationship between the fibre angle and dynamic response of the beam under harmonic excitation, indicating an increase of maximum structural displacement as the fibre angle was increased. The proposed modelling method can therefore be used for analysing the forced vibration characteristics of composite laminate beams with different fibre angles. It can also be used to optimize the design configuration of the composite beams for a variety of applications.

5. ACKNOWLEDGEMENTS

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